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REMARKS ON TRANSIENT LAMINAR FREE CONVECTION ALONG A VERTICAL PLATE*

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INTRODUCTION

FOR TRANSIENT laminar free convection along a vertical plate, which undergoes step increase in temperature or heat flux from an isothermal field, it is now well recognized that one-dimensional conduction effects prevail in the early stages of the transient period, while the leading-edge effect begins to be felt in an increasing degree later, eventually leading to asymptotic steady-state behavior as time approaches infinity. Analytically, this problem has not been treated to a very satisfactory degree in the literature. The integral solution of Siegel [1], though covering the complete Prandtl number range and both cases of step increase in temperature and heat flux, suffers the usual uncertainty in accuracy, and also exhibits unrealistic discontinuities in the transient periods. The purely numerical solution of Hellums and Churchill [2] remains the only complete continuous solution to this problem. Unfortunately, their results are only for the case of step change in surface temperature and one Prandtl number of 0·733. Very recently, Goldstein and Briggs [3] have found it desirable in certain experimental situations to determine the time instants in the transient period at which the purely one-dimensional conduction effects end, and have provided a criterion for this purpose with essentially no justification. Briefly, it is based on considering at each instant a penetration distance beyond which the conduction solution is valid, and this penetration distance is taken to be the maximum distance the leading-edge effect propagates along the plate with a velocity as given by the conduction solution. It seems questionable that the penetration velocity of the leading-edge effect, which is primarily related to the leading-edge phenomenon, should be determined from the conduction solution, which is entirely independent of the leading edge. Consequently, this point must be cleared up before their quantitative results can be put to general use, despite the fact that their results do compare well with some experimental data, and somewhat less so with the numerical result of Hellums and Churchill [2]. The purpose of the present study is not to attempt a direct analytical solution to this difficult problem, but to re-examine from a mathematical viewpoint the problem of determining the range of validity of the pure conduction region. It will be seen that the end of this conduction region is closely related to the possible occurrence of essential singularity in the governing differential equation. A criterion can then be obtained, which is shown to be reducible to that proposed by Goldstein and Briggs [3], thus giving their quantitative results a needed justification. The present approach follows closely the mathematical study of Stewartson [4] on the problem of impulse flow over a semi-infinite flat plate.

FORMULATION AND ANALYSIS

Laminar boundary-layer equations for unsteady free convection along a semi-infinite vertical plate with constant properties except slight changes in density and negligible viscous dissipation are well known, and may be readily written in the following dimensionless forms:

$$u_t + uu_x + vu_y = G + u_{yy} \tag{1}$$

$$u_x + v_y = 0 \tag{2}$$

$$G_t + uG_x + vG_y = (1/\sigma)\, G_{yy} \tag{3}$$

where $x = \bar{x}/L$, $y = \bar{y}/L$, $t = \nu\bar{t}/L^2$, $u = \bar{u}L/\nu$, $v = \bar{v}L/\nu$, $G = g\beta L^3(\bar{T} - \bar{T}_\infty)/\nu^2$ and $\bar{x}$ and $\bar{y}$ are the physical coordinates shown in Fig. 1, $\bar{u}$ and $\bar{v}$ the velocity components in

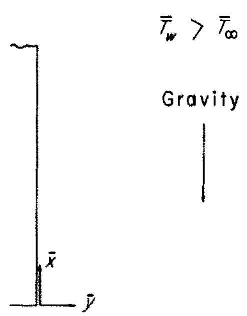

FIG. 1. Space coordinate system.

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the $\bar{x}$- and $\bar{y}$-directions, respectively, $\bar{T}$ the temperature variable, $\bar{t}$ the time variable, g the gravitational acceleration, β the coefficient of volumetric expansion, ν the kinematic viscosity, L the plate height and σ the Prandtl number. Subscripts x, y and t denote respective partial derivatives, while subscript ∞ refers to undisturbed ambient condition. For the transient problem under consideration, the initial and boundary conditions are:

$$t \leqslant 0 \qquad u = G = 0 \tag{4}$$

$$\left.\begin{array}{llll} t > 0 & y = 0 & u = v = 0 & G = a \text{ or } G_y = -q_w \\ & y \to \infty & u \to 0 & G \to 0 \end{array}\right\} \tag{5}$$

where the dimensionless surface temperature a and surface heat flux q_w are both taken to be constant. The general qualitative behavior of the solution to the above equations is known. At small times, one-dimensional transient conduction prevails, and both G and u are only functions of y and t. The x-dependency of the solutions, or the leading-edge effect, does not occur until a specific combination of x and t is reached, marking the end of the conduction region. Thereafter, the leading-edge effect becomes increasingly more dominant and finally the steady-state similarity solutions are approached when t becomes large. To bring out these limiting behaviors as well as mathematical difficulties associated with the present problem, it is desirable to re-cast the dependent variables in equations (1)–(3) in the following forms:

$$\begin{aligned} \psi(x, y, t) &= t^{3/2}\, G_w(t)\, F(\tau, Y) \\ G(x, y, t) &= G_w(t)\, \theta(\tau, Y) \end{aligned} \tag{6}$$

where ψ is the usual stream function such that $u = \psi_y$ and $v = -\psi_x$, and $G_w(t)$ is the plate surface temperature expressed in the form $G_w(t) = at^p$, introduced here for the purpose of unifying the analyses for both cases of step change in surface temperature and heat flux. As shown by Menold and Yang [5], $p = 0$ is for the case of step increase in temperature and $p = \frac{1}{2}$ signifies the corresponding heat-flux case. The two new independent variables $\tau = t^2 G_w(t)/x$ and $Y = y/(2\sqrt{t})$ now replace x, y and t in the original equations. Based on equations (1)–(3), it may be readily shown that the new dependent variables F and θ are now governed by the following respective equations:

$$\begin{aligned} &F_{YYY} + 4YF_{YY} - 4(1 + p)\,F_Y + 8\theta \\ &\qquad = [(4p + 8)\,\tau - 2\tau^2 F_Y]\,F_{\tau Y} + 2\tau^2 F_\tau F_{YY} \end{aligned} \tag{7}$$

$$\begin{aligned} &(1/\sigma)\,\theta_{YY} + 2Y\theta_Y - 4p\theta \\ &\qquad = [(4p + 8)\,\tau - 2\tau^2 F_Y]\,\theta_\tau + 2\tau^2 F_\tau \theta_Y \end{aligned} \tag{8}$$

where subscripts τ and Y again denote partial derivatives. The initial and boundary conditions, according to (4) and (5), now reduce to

$$\left.\begin{array}{l} F(0, Y) = F_0(Y) \qquad \theta(0, Y) = \theta_0(Y) \\ F(\tau, 0) = F_Y(\tau, 0) = F_Y(\tau, \infty) = \theta(\tau, \infty) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \theta(\tau, 0) = 1 \end{array}\right\} \tag{9}$$

where $F_0(Y)$ and $\theta_0(Y)$ are the pure conduction solutions satisfying equations (7) and (8) with $\tau = 0$. They have already been given by Menold and Yang [5] and also by Schetz and Eichhorn [6] for the complete range of Prandtl numbers, and hence will not be given here.

Mere inspection of equations (7) and (8) may suggest solutions in the form of asymptotic series expansions

$$F(\tau, Y) = \sum_{n=0,1,\ldots}^{\infty} \tau^n F_n(Y) \qquad \theta(\tau, Y) = \sum_{n=0,1,\ldots}^{\infty} \tau^n \theta_n(Y) \tag{10}$$

Unfortunately, this scheme fails completely in view that it may be readily shown that all functions F_n and θ_n for $n \neq 0$ vanish identically, when equations (10) are substituted in (7) and (8) and terms of like powers of τ are collected. Mathematically, the failure of this approach strongly indicates the possible existence of an essential singularity in the solution at a certain value of τ, at which the dependent functions are identical to the τ-independent solution, and there, all derivatives of these functions with respect to τ are zero. It is perhaps evident that this value of τ represents the end of the pure conduction region. The strong possibility of such an essential singularity to exist is also well substantiated by the physical phenomenon itself. Initially, the free-convection process is independent of x and a finite interval of time must elapse before the leading-edge effect can be felt at a given x. This transition can only be carried out through the presence of an essential singularity.

To determine the exact location of this singularity is still rather difficult, since the mathematical theory on the occurrence of essential singularities in differential equations has not been developed to any extent even at the present time. However, it is known that they usually occur when the coefficients of the leading derivative terms in the differential equation vanish. Since in the neighborhood of this singularity, F and θ are expected to deviate, at best, only slightly from F_0 and θ_0, respectively, it is here only necessary to consider small deviations from the conduction solutions. Thus, let

$$F(\tau, Y) = F_0(Y) + \bar{F}(\tau, Y) \qquad \theta(\tau, Y) = \theta_0(Y) + \bar{\theta}(\tau, Y) \tag{11}$$

where $\bar{F} \ll F_0$ and $\bar{\theta}_0 \ll \theta_0$. When these are substituted into equations (7) and (8) and higher orders of $\bar{F}$ and $\bar{\theta}$ are neglected, we obtain

$$\begin{aligned} &\bar{F}_{YYY} + 4Y\bar{F}_{YY} - 4(1 + p)\,\bar{F}_Y + 8\bar{\theta} \\ &\qquad = [(4p + 8)\,\tau - 2\tau^2 F_0']\,\bar{F}_{\tau Y} + 2\tau^2 F_0''\bar{F}_\tau \end{aligned} \tag{12}$$

$$\begin{aligned} &(1/\sigma)\,\bar{\theta}_{YY} + 2Y\bar{\theta}_Y - 4p\bar{\theta} \\ &\qquad = [(4p + 8)\,\tau - 2\tau^2 F_0']\,\bar{\theta}_\tau + 2\tau^2 \theta_0 \bar{F}_\tau \end{aligned} \tag{13}$$

The only coefficient of the leading derivative terms that can possibly be zero is $[(4p + 8)\tau - 2\tau^2 F_0']$, which vanishes when (1) $\tau = 0$, (2) $\tau = (2p + 4)/F_{0m}$ or (3) $\tau = (2p + 4)/F_0'$ for $F_0' < F_{0m}'$, where F_{0m}' is the maximum value of F_0', which depends only on p and σ. In addition to these three possibilities, the singularity could also occur anywhere between $0 < \tau < (2p + 4)/F_{0m}'$. Here primes all refer to derivatives with respect to Y.

The first possibility that the singularity occurs at $\tau = 0$ can be immediately ruled out, since the terms reflecting the leading-edge effect are all of the order of τ^2, and yet the solution sought is of the order of τ. The likelihood of having an essential singularity in the region $0 < \tau < (2p + 4)/F_{0m}'$ may also be ruled out, based on the following considerations. In this region, the coefficient $[(4p + 8)\tau - 2\tau^2 F_0']$ is nonzero and positive, and for the present purpose it is desirable to simplify equations (12) and (13) by retaining only the leading derivative terms with respect to τ and letting this nonvanishing coefficient be replaced by a positive constant γ. For simplifying the presentation, only equation (13) is considered here. Thus, we have

$$(1/\sigma)\,\bar{\theta}_{YY} + 2Y\bar{\theta}_Y - 4p\bar{\theta} = \gamma\bar{\theta}_\tau \tag{14}$$

with the conditions $\tau = \varepsilon, \bar{\theta} = 0$; $Y = 0, \bar{\theta} = 0$; and $Y \to \infty$, $\bar{\theta} \to 0$, where ε is the location of the essential singularity, if it exists in this region. The above equation has a solution satisfying the two boundary conditions in the form

$$\bar{\theta} = \int_0^\infty \exp\left[-\frac{s^2}{4\sigma}\right] s^{-(1+2p)}\,\phi\left\{s \exp\left[\frac{2}{\gamma}(\varepsilon - \tau)\right]\right\} \times \sin sY \, ds$$

where ϕ is an arbitrary function. The initial condition $\tau = \varepsilon$, $\bar{\theta} = 0$ immediately leads to $\phi = 0$, and consequently $\bar{\theta} \equiv 0$ for any Y and τ in this region. Similarly, it can be shown that F is also identically zero. Consequently, in this region no essential singularity is likely to exist.

This leads us to the next possible location at $\tau = \tau_c$, where $\tau_c = (2p + 4)/F_{0m}'$. At the present time, there is no conclusive evidence that an essential singularity must occur at $\tau = \tau_c$. The chief difficulty lies in that uniqueness theorem of solutions to the laminar boundary-layer equations is not yet proven. Hence, even if we could obtain a solution which possesses an essential singularity at $\tau = \tau_c$, there is still no guarantee of its existence. For this reason, such a solution is not attempted here. However, this value τ_c, according to the previous analysis, does represent the minimum possible value. From the literature, there are a number of indications which do suggest that τ_c is what we are seeking. The experimental data of Goldstein and Eckert [7], the integral solutions of Siegel [1] and Gebhart [8], and the numerical solution of Hellums and Churchill [2] all indicate the end of pure conduction region occurring in the general neighborhood of τ_c. Consequently, the region of validity of the one-dimensional transient conduction solution can be placed in $0 \leqslant \tau \leqslant \tau_c$.

Now we are in a position to compare this criterion with that somewhat arbitrarily chosen by Goldstein and Briggs [3]. As pointed out previously, their assertion is that the leading-edge effect propagates along the plate with a velocity as given by the x-independent conduction solution. The maximum penetration distance x at time t, beyond which the conduction solution is valid, is then used as a criterion for locating the end of the conduction region. More specifically, it may be written, in the present notation, as

$$x = \left[\int_0^t u\,dt\right]_{max}$$

which reduces to

$$x = \left[\int_0^t \frac{tG_w}{2} F_0'\,dt\right]_{max} = F_{0m}' \int_0^t \frac{tG_w}{2}\,dt$$

or

$$\frac{t^2 G_w}{x} = \frac{2(p + 2)}{F_{0m}'}$$

which is seen to be identical to τ_c. Consequently, all quantitative results obtained by Goldstein and Briggs [3] for the cases of step increase in surface temperature and in surface heat flux, relative to the extent of the pure conduction region, are now justifiable.

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